Isometric Surfaces and the Third Laplace-Beltrami

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ABSTRACT. In this paper, classical isometric helicoidal and rotational surfaces are studied, and generalized by Bour's theorem in three dimensional Euclidean space. Moreover, the third Laplace-Beltrami operators of two classical surfaces are obtained.

1. Introduction

The right helicoid (resp. catenoid) is the only ruled (resp. rotational (or surface of revolution)) surface which is minimal in classical surface geometry in Euclidean space. If we focus on the ruled (helicoid) and rotational characters, we have Bour's theorem in [**1, 3**].

Ikawa determined pairs of surfaces by Bour's theorem with the additional condition that they have the same Gauss map in Euclidean 3-space in [**3**]. Some relations among the Laplace-Beltrami operator and curvatures of the helicoidal surfaces in Euclidean 3-space are shown by the present authors and Hacısalihoğlu in [**2**]. In addition, they give Bour's theorem on the Gauss map, and some special examples.

We recall some basic notions of the Euclidean geometry, and the reader can be found the definitions of helicoidal and rotational surfaces in section 2. In section 3, Isometric general helicoidal and rotational surfaces are obtained by Bour's theorem. Properties of the general isometric surfaces which have the same Gauss map and the minimality are investigated. Finally, the third Laplace-Beltrami operators of two surfaces are studied in section 4.

2. Preliminaries

In the rest of this paper we shall identify a vector (a,b,c) with its transpose $(a,b,c)^t$. In this section, we will give the rotational and helicoidal surfaces in Euclidean 3-space $\mathbb{E}^3$.

For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a curve in a plane Π in $\mathbb{E}^3$, and let ℓ be a straight line in Π. A *rotational surface* in $\mathbb{E}^3$ is defined as a surface rotating a curve γ around a line ℓ (these are called the *profile curve* and the *axis*, respectively). Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines orthogonal to the axis ℓ, so that the speed of

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displacement is proportional to the speed of rotation. Then the resulting surface is called the *helicoidal surface* with axis ℓ and pitch $a \in \mathbb{R} \setminus \{0\}$.

We may suppose that ℓ is the line spanned by the vector $(0,0,1)$. The orthogonal matrix which fixes the above vector is

$$A(v) = \begin{pmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v \in \mathbb{R}.$$

The matrix A can be found by solving the following equations simultaneously; $A\ell = \ell$, $A^t A = AA^t = I_3$, $\det A = 1$. When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the z-axis of $\mathbb{E}^3$. Parametrization of the profile curve is given by $\gamma(u) = (\zeta(u), 0, \varphi(u))$, where $\zeta(u), \varphi(u) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable functions for all $u \in I$. A helicoidal surface in three dimensional Euclidean space which is spanned by the vector $(0,0,1)$ with pitch a, as follow

$$\mathbf{H}(u,v) = A(v)\ \gamma(u) + av\ell.$$

When $a = 0$, the surface is just a rotational surface as follow

$$\mathbf{R}(u,v) = (\zeta(u)\cos v, \zeta(u)\sin v, \varphi(u)).$$

3. Isometric surfaces

In this section, the rotational surface that is isometric to the helicoidal surface will be generalized by Bour's theorem in three dimensional Euclidean space.

Theorem 3.1. *A helicoidal surface*

$$\mathbf{H}(u,v) = (\zeta(u)\cos v, \zeta(u)\sin v, \varphi(u) + av) \tag{3.1}$$

is isometric to a rotational surface

$$\mathbf{R}(u,v) = \begin{pmatrix} \sqrt{\zeta^2 + a^2}\cos\left(v + \int \frac{a\varphi'}{\zeta^2+a^2}du\right) \\ \sqrt{\zeta^2 + a^2}\sin\left(v + \int \frac{a\varphi'}{\zeta^2+a^2}du\right) \\ \int \sqrt{\frac{(a\zeta')^2+(\zeta\varphi')^2}{\zeta^2+a^2}}du \end{pmatrix} \tag{3.2}$$

by Bour's theorem, where ζ and φ are differentiable functions, $0 \le v \le 2\pi$ and $u, a \in \mathbb{R} \setminus \{0\}$.

Proof. We assume that the profile curve on the helicoidal surface is $\gamma(u) = (\zeta(u), 0, \varphi(u))$. Since the helicoidal surface is given by rotating the profile curve γ around the axis $\ell = (0,0,1)$ and simultaneously displacing parallel lines orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation, we have the following representation of the helicoidal surface

$$\mathbf{H}(u,v) = (\zeta(u)\cos v, \zeta(u)\sin v, \varphi(u) + av),$$

where $u, a \in \mathbb{R} \setminus \{0\}$ and $0 \le v \le 2\pi$. The line element of the helicoidal surface as above is given by

$$ds^2 = \left(\zeta'^2 + \varphi'^2\right)du^2 + 2a\varphi' du dv + \left(\zeta^2 + a^2\right)dv^2. \tag{3.3}$$

Helices in $\mathbf{H}(u,v)$ are curves defined by $u = const.$, so curves in $\mathbf{H}(u,v)$ that are orthogonal to helices supply the orthogonality condition $2a\varphi' du + \left(\zeta^2 + a^2\right)dv = 0$. Thus, we obtain $v = -\int \frac{a\varphi'}{\zeta^2+a^2}du + c$, where c is constant. Hence if we put

$\overline{v} = v + \int \frac{a\varphi'}{\zeta^2+a^2} du$, then curves orthogonal to helices are given by $\overline{v} = const.$. Substituting the equation $dv = d\overline{v} - \frac{a\varphi'}{\zeta^2+a^2} du$ into the line element (3.3), we have

$$ds^2 = \left(\zeta'^2 + \frac{\zeta^2\varphi'^2}{\zeta^2+a^2}\right) du^2 + \left(\zeta^2 + a^2\right) d\overline{v}^2. \tag{3.4}$$

Setting $\overline{u} := \int \sqrt{\left(\zeta'^2 + \frac{\zeta^2\varphi'^2}{\zeta^2+a^2}\right)} du$, $k\left(\overline{u}\right) := \sqrt{\zeta^2 + a^2}$, (3.4) becomes

$$ds^2 = d\overline{u}^2 + k^2\left(\overline{u}\right) d\overline{v}^2. \tag{3.5}$$

The rotational surface

$$\mathbf{R}(u_{\mathbf{R}}, v_{\mathbf{R}}) = \left(\zeta_{\mathbf{R}}\left(u_{\mathbf{R}}\right)\cos v_{\mathbf{R}}, \zeta_{\mathbf{R}}\left(u_{\mathbf{R}}\right)\sin v_{\mathbf{R}}, \varphi_{\mathbf{R}}\left(u_{\mathbf{R}}\right)\right) \tag{3.6}$$

has the line element

$$ds_{\mathbf{R}}^2 = \left(\zeta_{\mathbf{R}}'^2 + \varphi_{\mathbf{R}}'^2\right) du_{\mathbf{R}}^2 + \zeta_{\mathbf{R}}^2 d\overline{v}_{\mathbf{R}}^2. \tag{3.7}$$

Again, setting $\overline{u}_{\mathbf{R}} := \int \sqrt{\zeta_{\mathbf{R}}'^2 + \varphi_{\mathbf{R}}'^2} du_{\mathbf{R}}$, $k_{\mathbf{R}}\left(\overline{u}_{\mathbf{R}}\right) := \zeta_{\mathbf{R}}$, then (3.7) becomes

$$ds_{\mathbf{R}}^2 = d\overline{u}_{\mathbf{R}}^2 + k_{\mathbf{R}}^2\left(\overline{u}_{\mathbf{R}}\right) d\overline{v}_{\mathbf{R}}^2. \tag{3.8}$$

Comparing (3.5) with (3.8), if we take $\overline{u} = \overline{u}_{\mathbf{R}}$, $\overline{v} = \overline{v}_{\mathbf{R}}$, $k\left(\overline{u}\right) = k_{\mathbf{R}}\left(\overline{u}_{\mathbf{R}}\right)$, then we have an isometry between $\mathbf{H}(u,v)$ and $\mathbf{R}(u_{\mathbf{R}}, v_{\mathbf{R}})$. Therefore, it follows that

$$\int \sqrt{\left(\zeta'^2 + \frac{\zeta^2\varphi'^2}{\zeta^2+a^2}\right)} du = \int \sqrt{\zeta_{\mathbf{R}}'^2 + \varphi_{\mathbf{R}}'^2} du_{\mathbf{R}},$$

and we can see

$$\varphi_{\mathbf{R}} = \int \sqrt{\frac{\left(a\zeta'\right)^2 + \left(\zeta\varphi'\right)^2}{\zeta^2 + a^2}} du.$$

Figure 1. A helicoidal surface

Figure 2. A rotational surface

Example 3.1. A helicoidal surface (see Fig. 1)

$$H\left(u,v\right) = \left(u^2\cos v, u^2\sin v, u^3 + av\right)$$

is isometric to the rotational surface

$$R(u,v)=\begin{pmatrix} \sqrt{u^4+a^2}\cos\left(v+\int\frac{3au^2}{u^4+a^2}du\right) \\ \sqrt{u^4+a^2}\sin\left(v+\int\frac{3au^2}{u^4+a^2}du\right) \\ \int\sqrt{\frac{4a^2u^2+9u^8}{u^4+a^2}}du \end{pmatrix}$$

by Bour's Theorem, where $u,a\in\mathbb{R}\setminus\{0\}$ and $0\leq v\leq 2\pi$. Moreover, when $a=0$, these surfaces has the form of the rotational surface (see Fig. 2) $\left(u^2\cos v, u^2\sin v, u^3\right)$.

Corollary 3.2. *If $I:\mathbf{H}(u,v)\longrightarrow\mathbf{R}(u,v)$ is an isometry (and surfaces are locally isometric), then the Gausssian curvatures at corresponding points are equal, and*

$$K_{\mathbf{H}}(p)=\frac{\zeta^3\varphi'^2\zeta''-\zeta^3\zeta'\varphi'\varphi''-a^2\zeta'^4}{\left(\left(\zeta^2+a^2\right)\zeta'^2+\zeta^2\varphi'^2\right)^2}=K_{\mathbf{R}}(I(p))$$

for all point p in $\mathbf{H}$.

Now, we prove the following theorem if the isometric surfaces have the same Gauss map.

Theorem 4.1. *Let a helicoidal and a rotational surface be isometrically related by Bour's theorem. If these two surfaces have the same Gauss map, then a pair of two surfaces is*

$$\mathbf{H}(u,v)=(\zeta(u)\cos v,\zeta(u)\sin v,\varphi(u)+av),$$

and

$$\mathbf{R}(u,v)=\begin{pmatrix} \sqrt{\zeta^2+a^2}\cos\left(v+\int\frac{a\varphi'}{\zeta^2+a^2}du\right) \\ \sqrt{\zeta^2+a^2}\sin\left(v+\int\frac{a\varphi'}{\zeta^2+a^2}du\right) \\ b\arg\cosh\left(\frac{\sqrt{\zeta^2+a^2}}{b}\right) \end{pmatrix}, \tag{3.9}$$

where

$$\begin{aligned}\varphi(u) &= -\frac{rb^4}{3\sqrt{1-t^2}\sqrt{a^2+r^2t^2}\left(r^2+a^2\right)^3}[2r^2(r^2-3a^2)t^4 \\ &\quad +(-3r^4+10a^2r^2-3a^4)t^2+2a^2(-3r^2+a^2)],\end{aligned}$$

$t:=\sqrt{\frac{\zeta^2+a^2}{\zeta^2+a^2-b^2}}$, $r:=\sqrt{b^2-a^2}$, $\zeta=\zeta(u)$ is a differentiable function, $u,a,b\in R\setminus\{0\}$, and $0\leq v\leq 2\pi$.

Proof. First we consider the helicoidal surface (3.1). By virtue of the coefficients of the first and second fundamental forms $E=\zeta'^2+\varphi'^2$, $F=a\varphi'$, $G=\zeta^2+a^2$, $L=\zeta(-\varphi'\zeta''+\zeta'\varphi'')(\det\mathbf{I})^{-1/2}$, $M=-a\zeta'^2(\det\mathbf{I})^{-1/2}$, $N=-\zeta^2\varphi'(\det\mathbf{I})^{-1/2}$, the Gauss map and the mean curvature of the helicoidal surface are

$$e=\frac{1}{\sqrt{\det\mathbf{I}}}\begin{pmatrix} a\zeta'\sin v-\zeta\varphi'\cos v \\ -a\zeta'\cos v-\zeta\varphi'\sin v \\ \zeta\zeta' \end{pmatrix}, \tag{3.10}$$

$$H=\frac{\Phi(u)}{2(\det\mathbf{I})^{3/2}}, \tag{3.11}$$

where det$\mathbf{I}$= $\left(\zeta^2+a^2\right)\zeta'^2+\zeta^2\varphi'^2$,

$$\Phi(u):=\left(\zeta^2\zeta'^2-\zeta^3\zeta''-a^2\zeta\zeta''+2a^2\zeta'^2\right)\varphi'+\zeta^2\varphi'^3+\left(\zeta^3\zeta'+a^2\zeta\zeta'\right)\varphi''. \tag{3.12}$$

Next, we calculate the Gauss map $e_{\mathbf{R}}$ and the mean curvature $H_{\mathbf{R}}$ of the rotational surface (3.2). The Gauss map and the mean curvature of the rotational surface, respectively, is

$$e_{\mathbf{R}}=\frac{1}{\sqrt{\det\mathbf{I}}}\begin{pmatrix}\sqrt{\left(a\zeta'\right)^2+\left(\zeta\varphi'\right)^2}\cos\left(v+\int\frac{a\varphi'}{\zeta^2+a^2}du\right)\\ \sqrt{\left(a\zeta'\right)^2+\left(\zeta\varphi'\right)^2}\sin\left(v+\int\frac{a\varphi'}{\zeta^2+a^2}du\right)\\ \zeta\zeta'\end{pmatrix}, \tag{3.13}$$

$$H_{\mathbf{R}}=\frac{\zeta^2\varphi'\Phi(u)}{2\sqrt{\zeta^2+a^2}\sqrt{\left(a\zeta'\right)^2+\left(\zeta\varphi'\right)^2}\left(\det\mathbf{I}\right)^2}. \tag{3.14}$$

Now, suppose that the Gauss map e is identically equal to $e_{\mathbf{R}}$. If $\varphi'=0$, then the helicoidal surface reduces to right helicoid and the mean curvature of the rotational surface is identically zero. Hence, the rotational surface is the catenoid and the function $\varphi_{\mathbf{R}}(u_{\mathbf{R}})$ of (3.6) is $\varphi_{\mathbf{R}}(u_{\mathbf{R}})=b\operatorname{arg\,cosh}\left(\frac{u_{\mathbf{R}}}{b}\right)$, where b is a constant. Comparing this equation and the third element of (3.2), we have $b\operatorname{arg\,cosh}\left(\frac{\sqrt{\zeta^2+a^2}}{b}\right)=\int\frac{a\zeta'}{\sqrt{\zeta^2+a^2}}du$. By differentiating this equation, it follows that $\frac{b\zeta}{\sqrt{\zeta^2+a^2-b^2}}=a$. Therefore, we have $a=b$. Next, we suppose $\varphi'\neq 0$. Comparing e and $e_{\mathbf{R}}$, we have $\operatorname{arg\,tanh}\left(\frac{a\zeta'}{\zeta\varphi'}\right)=\int\frac{a\varphi'}{\zeta^2+a^2}du$. Differentiating this equation, we obtain

$$\left(\zeta^2\zeta'^2-\zeta^3\zeta''-a^2\zeta\zeta''+2a^2\zeta'^2\right)\varphi'+\zeta^2\varphi'^3+\left(\zeta^3\zeta'+a^2\zeta\zeta'\right)\varphi''=0. \tag{3.15}$$

This equation means $\Phi(u)=0$ in (3.11) and (3.14). So, the helicoidal surface and the rotational surface are minimal surfaces. Hence, again, the rotational surface reduces to the catenoid. Then, it follows that

$$b\operatorname{arg\,cosh}\left(\frac{\sqrt{\zeta^2+a^2}}{b}\right)=\int\sqrt{\frac{\left(a\zeta'\right)^2+\left(\zeta\varphi'\right)^2}{\zeta^2+a^2}}du.$$

Using this eqution, we can find the profile curve φ of the helicoidal surface. Then, we have

$$\varphi'=\frac{\sqrt{b^2-a^2}\sqrt{\zeta^2+a^2}\zeta'}{\zeta\sqrt{\zeta^2+a^2-b^2}}.$$

We put $t:=\sqrt{\frac{\zeta^2+a^2}{\zeta^2+a^2-b^2}}$ and $r:=\sqrt{b^2-a^2}$, then it follows that

$$\varphi=r\int\frac{b^4t^3}{\left(r^2t^2+a^2\right)^{3/2}\left(1-t^2\right)^{5/2}}dt.$$

4. The Third Laplace Beltrami Operator

In this section, we will study on the third Laplace-Beltrami operators of well known classical isometric minimal surfaces.

Let $x = x(u^1, u^2)$ be a surface of 3−dimensional Euclidean space defined in D. The same for the functions ϕ, ψ. Let $n = n(u^1, u^2)$ be the normal vector of the surface. We write

$$g_{ij} = \langle x_i, x_j \rangle\,,\ b_{ij} = \langle x_{ij}, n \rangle\,,\ e_{ij} = \langle n_i, n_j \rangle\,. \tag{4.1}$$

The equations of Weingarten are

$$\begin{aligned} x_i &= b_{ij} e^{jr} n_r = -g_{ij} b^{jr} n_r, \\ n_i &= -e_{ij} b^{jr} x_r = -b_{ij} g^{jr} x_r, \end{aligned}$$

where $x_i = \frac{\partial x}{\partial u^i}$. Then the first parameter Beltrami is defined

$$grad^{\mathbf{III}}\left(\phi,\psi\right) = e_i^{ik} \phi \psi_k.$$

Using following expressions

$$grad^{\mathbf{III}}\left(\phi\right) = grad^{\mathbf{III}}\left(\phi,\phi\right) = e^{ik} \phi_i \psi_k,$$

$$grad^{\mathbf{III}} \phi = grad^{\mathbf{III}}\left(\phi,n\right) = e^{ik} \phi_i n_k,$$

the second parameter Beltrami is defined

$$\Delta^{\mathbf{III}} \phi = -e^{ik} grad_k^{\mathbf{III}} \phi_i.$$

Using the last relation we get the expression the third Laplace-Beltrami operator of the function ϕ. So, we have the third fundamental form (see [**4**] for details) as follow

$$\Delta^{III} \phi = -\frac{\sqrt{\det \mathbf{I}}}{\det \mathbf{II}} \left[\frac{\partial}{\partial u} \left(\frac{Z\phi_u - Y\phi_v}{\sqrt{\det \mathbf{I}} \det \mathbf{II}} \right) - \frac{\partial}{\partial v} \left(\frac{Y\phi_u - X\phi_v}{\sqrt{\det \mathbf{I}} \det \mathbf{II}} \right) \right], \tag{4.2}$$

where $u = u^1, v = u^2$, the coefficients of the first (resp., second, and third) fundamental form of the function ϕ is E, F, G (resp., L, M, N, and X, Y, Z), $\det \mathbf{I} = EG - F^2$, $\det \mathbf{II} = LN - M^2$, $X = EM^2 - 2FLM + GL^2$, $Y = EMN - FLN + GLM - FM^2$, $Z = GM^2 - 2FNM + EN^2$.

Let $\zeta\left(u\right) = u$ and $\varphi\left(u\right) = 0$ in Theorem 3.1, we have the well known classical isometric minimal surfaces. So, the right helicoid

$$H\left(u,v\right) = \left(u\cos v, u\sin v, av\right) \tag{4.3}$$

is isometric to the catenoid

$$R\left(u,v\right) = \left(\sqrt{u^2+a^2}\cos v, \sqrt{u^2+a^2}\sin v, a\log\left(u+\sqrt{u^2+a^2}\right)\right), \tag{4.4}$$

where $u, a \in \mathbb{R} \setminus \{0\}$, $0 \leq v \leq 2\pi$. The coefficients of the first and second fundamental forms of these surfaces are

$$E_{H(u,v)} = 1 = E_{R(u,v)},$$

$$F_{H(u,v)} = 0 = F_{R(u,v)},$$

$$G_{H(u,v)} = u^2 + a^2 = G_{R(u,v)},$$

$$L_{H(u,v)} = 0, \quad M_{H(u,v)} = -\frac{a}{\sqrt{u^2+a^2}}, \quad N_{H(u,v)} = 0,$$

$$L_{R(u,v)} = -\frac{a}{u^2+a^2}, \quad M_{R(u,v)} = 0, \quad N_{R(u,v)} = a.$$

Hence, we get

$$LN - M^2 = -\frac{a^2}{u^2+a^2}$$

for the two surfaces, and the Gaussian curvatures of the isometric minimal right helicoid and the catenoid are

$$K_{H(u,v)} = -\frac{a^2}{\left(u^2+a^2\right)^2} = K_{R(u,v)}.$$

Now, we apply the equation (4.2) for the right helicoid surface (4.3), and the catenoid surface (4.4). The coefficients of the third fundamental form of the surfaces are

$$\begin{gathered} X_{H(u,v)} = \frac{a^2}{u^2+a^2} = X_{R(u,v)}, \\ Y_{H(u,v)} = 0 = Y_{R(u,v)}, \\ Z_{H(u,v)} = a^2 = Z_{R(u,v)}. \end{gathered}$$

So, after some calculations for the surface $H = H(u,v)$ (resp. $R\,(u,v)$) we have

$$\begin{aligned} \left(\frac{Z\,(H_u) - Y\,(H_v)}{\sqrt{\det \mathbf{I} \det \mathbf{II}}}\right)_u &= \left(-\frac{u \sin v}{\sqrt{u^2+a^2}},\ -\frac{u \cos v}{\sqrt{u^2+a^2}},\ 0\right) \\ &= \left(\frac{Y\,(H_u) - X\,(H_v)}{\sqrt{\det \mathbf{I} \det \mathbf{II}}}\right)_v. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \Delta^{\mathbf{III}} H\,(u,v) &\equiv 0 \\ &\equiv \Delta^{\mathbf{III}} R\,(u,v)\,. \end{aligned}$$

This means, the surfaces are **III**$-$minimal.

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